THE FREIDLIN-WENTZELL LDP WITH RAPIDLY GROWING COEFFICIENTS

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Abstract. The Large Deviations Principle (LDP) is verified for a homogeneous diffusion process with respect to a Brownian motion B_t,

$$X_t^\varepsilon = x_0 + \int_0^t b(X_s^\varepsilon)ds + \varepsilon \int_0^t \sigma(X_s^\varepsilon)dB_s,$$

where $b(x)$ and $\sigma(x)$ are locally Lipschitz functions with super linear growth. We assume that the drift is directed towards the origin and the growth rates of the drift and diffusion terms are properly balanced. Nonsingularity of $a = \sigma\sigma^*(x)$ is not required.

1. Introduction

In this paper we extend the set of conditions, under which Freidlin-Wentzell's Large Deviation Principle (LDP) for a homogeneous diffusion process remains valid. We consider a family $\{(X_t^\varepsilon)_{t\ge 0}\}_{\varepsilon\to 0}$ of diffusions, where $X_t^\varepsilon \in \mathbb{R}^d$, $d \ge 1$ is defined by the Itô equation

$$X_t^\varepsilon = x_0 + \int_0^t b(X_s^\varepsilon)ds + \varepsilon \int_0^t \sigma(X_s^\varepsilon)dB_s, \tag{1.1}$$

with respect to a standard Brownian motion B_t, where $b(x)$ and $\sigma(x)$ are vector and matrix valued continuous functions of dimensions d and $d \times d$ respectively, guaranteeing existence of the unique weak solution.

The classical Freidlin-Wentzell setting [8] (see also Dembo and Zeitouni, [4]) is applicable to the model (1.1) with bounded $b(x)$ and $\sigma(x)$ and uniformly positive definite diffusion matrix $a(x) = \sigma\sigma^*(x)$. Various LDP versions can be found in Dupuis and Ellis [5], Feng [6], Feng and Kurtz [7], Friedman [9], Liptser and Pukhalskii [12], Mikami [15], Narita [16], Stroock [23], Ren and Zhang [22]. In the recent paper [19], Puhalskii extends LDP to (1.1) with continuous and unbounded coefficients and singular $a(x)$, assuming $b(x)$ and $a(x)$ are Lipschitz continuous functions (concerning singular $\sigma(x)$ see also Liptser et al, [14]). Being Lipschitz continuous, the entries of b, σ grow not faster than linearly and, thereby, *automatically* guarantee one of the necessary conditions for LDP ($\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$)

$$\lim_{C\to\infty} \varlimsup_{\varepsilon\to 0} \varepsilon^2 \log \mathsf{P}\Big(\sup_{t\le T} \|X_t^\varepsilon\| > C\Big) = -\infty, \ \forall\, T > 0. \tag{1.2}$$

Relinquishing the linear growth condition for b, σ would require additional assumptions providing (1.2).

This paper is inspired by Puhalskii's remark in [19]:

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If the drift is directed towards the origin, then no restrictions are needed on the growth rate of the drift coefficient.

In particular, in this case the LDP holds, regardless of the growth rate of $b(x)$, for a constant diffusion matrix (not necessarily nonsingular).

In this paper, we show that in fact LDP remains valid for (1.1) with non-constant diffusion term, if its growth rate is properly balanced relatively to the drift (see (H-3) of Theorem 2.1 below). Our result is formulated in terms of Khasminskii-Veretennikov's condition (H-2) (see [11] and [17], [18])

The rest of the paper is organized as follows. In Sections 2 and 3, the main result, notations and preliminary facts on the LDP are given. Sections 4 - 6 contain the proof of the main result. Auxiliary technical details are gathered in Appendices A - C.

2. **Notations and the main result**

The following notations and conventions are used through the paper.

- * denotes the transposition symbol
- all vectors are columns (unless explicitly stated otherwise)
- $|x|$ and $\|x\|$ denote the ℓ_1 and ℓ_2 (Euclidian) norms of $x \in \mathbb{R}^d$
- (x, y) denotes the scalar product of $x, y \in \mathbb{R}^d$
- $\|x\|^2_\Gamma = (x, \Gamma x)$ with an nonnegative definite matrix Γ
- $a(x) = \sigma(x)\sigma^*(x)$
- $a^\oplus(x)$ denotes the Moore-Penrose pseudoinverse matrix of $a(x)$ (see [1])
- $\nabla V(x)$ is the gradient (row) vector of $V(x)$:
$$\nabla V(x) := \Big(\frac{\partial V(x)}{\partial x_1}, \dots, \frac{\partial V(x)}{\partial x_d}\Big)$$
- $\langle M, N\rangle_t$ is the joint quadratic variation process of continuous martingales M_t and N_t; for brevity $\langle M, M\rangle_t = \langle M\rangle_t$
- a.s. abbreviates "almost surely"; when the corresponding measure is not specified the Lebesgue measure on $\mathbb{R}_+$ is understood
- ϱ is the locally uniform metric on $\mathbb{C}_{[0,\infty)}(\mathbb{R}^d)$
- $\mathbf{I}$ denotes $d \times d$ identity matrix
- the convention $0/0 = 0$ is kept throughout
- $X^\varepsilon = (X^\varepsilon_t)_{t\ge 0}$
- $\inf\{\varnothing\} = \infty$.

We study the LDP for the family $\{X^\varepsilon\}_{\varepsilon\to 0}$ in the metric space $(\mathbb{C}_{[0,\infty)}(\mathbb{R}^d), \varrho)$ with $\varrho(x, y) = \sum_{k=1}^\infty 2^{-k}\big(1 \vee \sup_{t\le k} \|x_t - y_t\|\big)$, $x, y \in \mathbb{C}_{[0,\infty)}(\mathbb{R}^d)$. Recall that $\{X^\varepsilon\}_{\varepsilon\to 0}$ satisfies the LDP with the good rate function $J(u) : \mathbb{C}_{[0,\infty)}(\mathbb{R}^d) \mapsto [0, \infty]$ and the rate ε^2, if the level sets of $J(u)$ are compact and for any closed set F and open set G in $\mathbb{C}_{[0,\infty)}(\mathbb{R}^d)$,

$$\varlimsup_{\varepsilon\to 0} \varepsilon^2 \log \mathsf{P}\big(X^\varepsilon \in F\big) \le -\inf_{u\in F} J(u),$$
$$\varliminf_{\varepsilon\to 0} \varepsilon^2 \log \mathsf{P}\big(X^\varepsilon \in G\big) \ge -\inf_{u\in G} J(u).$$

Our main result is

Theorem 2.1. *Assume*:

(H-1) *the entries of* $b(x)$ *and* $\sigma(x)$ *are locally Lipschitz continuous functions,*

(H-2) $\lim_{\|x\|\to\infty} \dfrac{(x,b(x))}{\|x\|} = -\infty$,

(H-3) *for some positive constants* K *and* L, $\dfrac{(x,a(x)x)}{\|x\|\ |(x,b(x))|} \le K$, $\forall\ \|x\| > L$.

Then $\{X^\varepsilon_t\}_{\varepsilon\to 0}$ *obeys the LDP in the metric space* $(\mathbb{C}_{[0,\infty)}(\mathbb{R}^d),\varrho)$ *with the rate* ε^2 *and the rate function*

$$J(u) = \begin{cases} \frac{1}{2}\int_0^\infty \|\dot u_t - b(u_t)\|^2_{a^\oplus(u_t)}dt, & u\in\varGamma \\ \infty, & u\notin\varGamma, \end{cases}$$

where

$$\varGamma = \left\{ u\in\mathbb{C}_{[0,\infty)} : \begin{matrix} u_0 = x_0,\ du_t \ll dt,\ \int_0^\infty \|\dot u_t\|^2 dt < \infty \\ a(u_t)a^\oplus(u_t)[\dot u_t - b(u_t)] = [\dot u_t - b(u_t)]\ a.s. \end{matrix} \right\}.$$

Remark 2.1. In the scalar case (recall 0/0=0)

$$J(u) = \begin{cases} \dfrac{1}{2}\int_0^\infty \dfrac{(\dot u_t - b(u_t))^2}{\sigma^2(u_t)}dt, & du_t = \dot u_t dt,\ u_0 = x_0,\ \int_0^\infty \dot u_t^2 dt < \infty \\ \infty, & \text{otherwise.} \end{cases}$$

Example 2.1. A typical example within the scope of Theorem 2.1 is

$$X^\varepsilon_t = x_0 - \int_0^t (X^\varepsilon_s)^3 ds + \varepsilon\int_0^t |X^\varepsilon_s|^{3/2} dB_s.$$

3. **Preliminaries**

We follow the framework, set up by A.Puhalskii (see [20], [21]):

$$\left.\begin{matrix}\text{Exponential tightness} \\ \text{Local LDP}\end{matrix}\right\} \Longleftrightarrow \text{LDP}$$

The exponential tightness in the metric space $(\mathbb{C}_{[0,\infty)},\varrho)$ is convenient to verify in terms of, so called, $\mathbb{C}$-*exponential tightness* conditions introduced by A.Puhalskii (see e.g. [12]), based on the stopping times technique introduced by D.Aldous in [2], [3]). To this end, let us assume that the diffusion processes are defined on a stochastic basis $(\varOmega,\mathcal{F},\mathbf{F}^\varepsilon = (\mathscr{F}^\varepsilon_t)_{t\ge 0},\mathsf{P})$, satisfying the usual conditions, where the filtration $\mathbf{F}^\varepsilon$ may depend on ε.

Recall (see [12]) that the family of diffusion processes is $\mathbb{C}$-exponentially tight if for any $T>0$, $\eta>0$ and any $\mathbf{F}^\varepsilon$-stopping time θ,

$$\lim_{C\to\infty}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le T}\|X^\varepsilon_t\| > C\Big) = -\infty, \tag{3.1}$$

$$\lim_{\triangle\to 0}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log\sup_{\theta\le T}\mathsf{P}\Big(\sup_{t\le\triangle}\|X^\varepsilon_{\theta+t} - X^\varepsilon_\theta\| > \eta\Big) = -\infty. \tag{3.2}$$

The family of diffusion processes obeys *the local LDP* in $(\mathbb{C}_{[0,\infty)}(\mathbb{R}^d),\varrho)$ if for any $T>0$ there exists a local rate function $J_T(u)$ such that

$$\varlimsup_{\delta\to 0}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le T}\|X^\varepsilon_t-u_t\|\le\delta\Big)\le -J_T(u) \tag{3.3}$$

$$\varliminf_{\delta\to 0}\varliminf_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le T}\|X^\varepsilon_t-u_t\|\le\delta\Big)\ge -J_T(u). \tag{3.4}$$

Under the conditions (3.1)-(3.4), the family of diffusion processes obeys the LDP with the rate ε^2 and the good rate function

$$J(u)=\sup_T J_T(u),\ u\in\mathbb{C}_{[0,\infty)}(\mathbb{R}^d),$$

where

$$J_T(u)=\begin{cases}\frac12\int_0^T\|\dot u_t-b(u_t)\|^2_{a^\oplus(u_t)}dt, & u\in\varGamma_T\\ \infty, & u\notin\varGamma_T,\end{cases}$$

with

$$\varGamma_T=\Big\{u\in\mathbb{C}_{[0,T]}:\ \begin{matrix}u_0=x_0,\ du_t\ll dt,\ \int_0^T\|\dot u_t\|^2dt<\infty\\ a(u_t)a^\oplus(u_t)[\dot u_t-b(u_t)]=[\dot u_t-b(u_t)]\ \text{a.s.}\end{matrix}\Big\}.$$

Thus the proof of Theorem 2.1 reduces to establishing (3.1) - (3.4).

4. The proof of $\mathbb{C}$-exponential tightness

4.1. **Auxiliary lemma.** Let $\mathfrak{D}$ be a nonlinear operator acting on continuously differentiable functions $V(x):\mathbb{R}^d\to\mathbb{R}$ as follows:

$$\mathfrak{D}V(x)=(\nabla V(x),b(x))+\frac12(\nabla V(x),a(x)\nabla V(x)).$$

Lemma 4.1. *Assume there exists twice continuously differentiable nonnegative function $V(x)$ such that*

(a-1) $\lim_{C\to\infty}\inf_{\|x\|\ge C}V(x)=\infty$

(a-2) *for some $L>0$, $\mathfrak{D}V(x)\le 0,\ \forall\ \|x\|>L$.*

Then (3.1) *holds.*

Proof. Notice that (3.1) is equivalent to

$$\lim_{C\to\infty}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\big(\varTheta_C\le T\big)=-\infty, \tag{4.1}$$

where

$$\varTheta_C=\inf\{t:\|X^\varepsilon_t\|\ge C\},\quad C>0 \tag{4.2}$$

are stopping times relative to $\mathbf{F}^\varepsilon$.

We use (a) of Proposition A.1 to estimate $\log\mathsf{P}(\varTheta_C\le T)$. An appropriate martingale M^ε_t is constructed with the help of function $V(x)$. Let $\Psi(x)$ be the Hessian of V, namely a matrix with the entries $V_{ij}(x)=\frac{\partial^2V(x)}{\partial x_i\partial x_j}$. By the Itô formula

$$\varepsilon^{-2}V(X^\varepsilon_{\varTheta_C\wedge t})=\varepsilon^{-2}V(x_0)+\int_0^{\varTheta_C\wedge t}\varepsilon^{-2}(\nabla V(X^\varepsilon_s),b(X^\varepsilon_s))ds$$
$$+\int_0^{\varTheta_C\wedge t}\varepsilon^{-1}(\nabla V(X^\varepsilon_s),\sigma(X^\varepsilon_s)dB_s)+\int_0^{\varTheta_C\wedge t}\frac12\operatorname{trace}\Big(\Psi(X^\varepsilon_s)a(X^\varepsilon_s)\Big)ds.$$

We choose $M^\varepsilon_t = \int_0^t \varepsilon^{-1}(\nabla V(X^\varepsilon_s), \sigma(X^\varepsilon_s)dB_s)$, which has the variation process $\langle M^\varepsilon\rangle_t = \int_0^t \varepsilon^{-2}(\nabla V(X^\varepsilon_s), a(X^\varepsilon_s)\nabla V(X^\varepsilon_s))ds$. Clearly

$$
\begin{aligned}
M^\varepsilon_{\Theta^\beta_C\wedge t} &= \varepsilon^{-2}V(X^\varepsilon_{\Theta_C\wedge t}) - \varepsilon^{-2}V(x_0)\\
&- \int_0^{\Theta_C\wedge t} \varepsilon^{-2}(\nabla V(X^\varepsilon_s), b(X^\varepsilon_s))ds - \int_0^{\Theta_C\wedge t} \frac{1}{2}\operatorname{trace}\Big(\Psi(X^\varepsilon_s)a(X^\varepsilon_s)\Big)ds.
\end{aligned}
$$

Hence, by the definition of $\mathfrak{D}$, one gets

$$
\begin{aligned}
M^\varepsilon_{\Theta_C\wedge T} - \frac{1}{2}\langle M^\varepsilon\rangle_{\Theta_C\wedge T} &= \varepsilon^{-2}V(X^\varepsilon_{\Theta_C\wedge T}) - \varepsilon^{-2}V(x_0)\\
&- \int_0^{\Theta_C\wedge T} \frac{1}{2}\operatorname{trace}\Big(\Psi(X^\varepsilon_s)a(X^\varepsilon_s)\Big)ds \quad - \int_0^{\Theta_C\wedge T} \varepsilon^{-2}\mathfrak{D}V(X^\varepsilon_s)ds. \qquad (4.3)
\end{aligned}
$$

On the set $\{\Theta_C \le T\}$, we have

$$
\varepsilon^{-2}V(X^\varepsilon_{\Theta_C\wedge T}) - \varepsilon^{-2}V(x_0) \ge \varepsilon^{-2}\inf_{\|x\|\ge C} V(x) - \varepsilon^{-2}V(x_0),
$$

and

$$
\Big|\int_0^{\Theta_C\wedge T} \frac{1}{2}\operatorname{trace}\big(\Psi(X^\varepsilon_s)a(X^\varepsilon_s)\big)ds\Big| \le \frac{T}{2}\sup_{\|x\|\le C}\big|\operatorname{trace}\big(\Psi(x)a(x)\big)\big|,
$$

and, by (a-2),

$$
\begin{aligned}
&- \int_0^{\Theta\wedge T} \varepsilon^{-2}\mathfrak{D}V(X^\varepsilon_s)ds\\
&\qquad \ge -\Big|\int_0^{\Theta_C\wedge T} \varepsilon^{-2}I_{\{\|X^\varepsilon_s\|\le L\}}\mathfrak{D}V(X^\varepsilon_s)ds\Big| \ge -\varepsilon^2 T\sup_{\|x\|\le L}|\mathfrak{D}V(x)|.
\end{aligned}
$$

These inequalities and (4.3) imply

$$
\begin{aligned}
M^\varepsilon_{\Theta_C} - \frac{1}{2}\langle M^\varepsilon\rangle_{\Theta_C} &\ge \varepsilon^{-2}\inf_{\|x\|\ge C} V(x) - \varepsilon^{-2}V(x_0)\\
&\qquad - \frac{T}{2}\sup_{\|x\|\le C}\big|\operatorname{trace}\big(\Psi(x)a(x)\big)\big| - \varepsilon^{-2}T\sup_{\|x\|\le L}\big|\mathfrak{D}V(x)\big|
\end{aligned}
$$

on the set $\{\Theta_C \le T\}$. Hence, due to (a) of Proposition A.1

$$
\begin{aligned}
&\varepsilon^2\log \mathsf{P}\big(\Theta_C \le T\big) \le\\
&- \inf_{\|x\|\ge C} V(x) + V(x_0) + \frac{T\varepsilon^2}{2}\sup_{\|x\|\le C}\big|\operatorname{trace}\big(\Psi(x)a(x)\big)\big| + T\sup_{\|x\|\le L}\big|\mathfrak{D}V(x)\big|\\
&\qquad\qquad \xrightarrow[\varepsilon\to 0]{} -\inf_{\|x\|\ge C} V(x) + V(x_0) + T\sup_{\|x\|\le L}\big|\mathfrak{D}V(x)\big|
\end{aligned}
$$

and it is left to recall that by (a-1) $\lim_{C\to\infty}\inf_{\|x\|\ge C} V(x) = \infty$. $\square$

4.2. **The proof of (3.1).** We apply Lemma 4.1 to

$$V(x)=\frac{c\|x\|^2}{1+\|x\|},$$

with a positive parameter $c\le\frac{1}{K}$ for K from (H-3) of Theorem 2.1. The function $V(x)$ is twice continuously differentiable and satisfies (a-1). It is left to show that $V(x)$ satisfies (a-2) as well.

Direct computations give $\nabla V(x)=c\frac{(2+\|x\|)\|x\|}{(1+\|x\|)^2}\frac{x}{\|x\|}$. Denote

$$r(x):=\frac{(2+\|x\|)\|x\|}{(1+\|x\|)^2}$$

and notice that $r(x)\le 1$. By assumption (H-2) of Theorem 2.1, one can choose $L>0$ sufficiently large so that $(x,b(x))<0$ for any $\|x\|\ge L$. On the other hand, by assumption (H-3) of Theorem 2.1, $-1+\frac{c}{2}\frac{(x,a(x)x)}{\|x\|\,|(x,b(x))|}\le-\frac{1}{2}$ for $\|x\|\ge L$ and

$$\begin{aligned}
\mathfrak{D}V(x)&=\left(c\frac{r(x)}{\|x\|}(x,b(x))+\frac{c^2r^2(x)}{2}\frac{(x,a(x)x)}{\|x\|^2}\right)\\
&=\left(-c\frac{r(x)}{\|x\|}\big|(x,b(x))\big|+\frac{c^2r^2(x)}{2}\frac{(x,a(x)x)}{\|x\|^2}\right)\\
&=cr(x)\frac{|(x,b(x))|}{\|x\|}\left(-1+\frac{c}{2}r(x)\frac{(x,a(x)x)}{\|x\|\,|(x,b(x))|}\right)\\
&\le cr(x)\frac{|(x,b(x))|}{\|x\|}\left(-1+\frac{c}{2}\frac{(x,a(x)x)}{\|x\|\,|(x,b(x))|}\right)\\
&\le-\frac{1}{2}cr(x)\frac{|(x,b(x))|}{\|x\|}
\end{aligned}$$

and (a-2) follows. □

4.3. **The proof of (3.2).** The obvious inclusion

$$\Big\{\sup_{t\le\triangle}\|X^\varepsilon_{\theta+t}-X^\varepsilon_\theta\|>\eta\Big\}\subseteq\Big\{\sup_{t\le\triangle}\|X^\varepsilon_{\theta+t}-X^\varepsilon_\theta\|>\eta,\ \varTheta_C=\infty\Big\}\bigcup\Big\{\varTheta_C\le T\Big\}$$

reduces the proof to verifying

$$\varlimsup_{\triangle\to0}\varlimsup_{\varepsilon\to0}\varepsilon^2\log\sup_{\theta\le T}\mathsf{P}\Big(\sup_{t\le\triangle}\|X^\varepsilon_{\theta+t}-X^\varepsilon_\theta\|>\eta,\ \varTheta_C=\infty\Big)=-\infty \tag{4.4}$$

for any fixed C. Indeed if (4.4) holds, then

$$\begin{aligned}
&\lim_{\triangle\to0}\varlimsup_{\varepsilon\to0}\varepsilon^2\log\sup_{\theta\le T}\mathsf{P}\Big(\sup_{t\le\triangle}\|X^\varepsilon_{\theta+t}-X^\varepsilon_\theta\|>\eta\Big)\\
&\le\lim_{\triangle\to0}\varlimsup_{\varepsilon\to0}\varepsilon^2\log\sup_{\theta\le T}\mathsf{P}\Big(\sup_{t\le\triangle}\|X^\varepsilon_{\theta+t}-X^\varepsilon_\theta\|>\eta,\ \varTheta_C=\infty\Big)\\
&\qquad\bigvee\varlimsup_{C\to\infty}\varlimsup_{\varepsilon\to0}\varepsilon^2\log\mathsf{P}(\varTheta_C\le T)
\end{aligned}$$

and, thus, (3.2) is implied by (4.4) and (4.1). So, it is left to check (4.4) for any entry x^ε_t of X^ε_t:

$$\lim_{\triangle\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2\log\sup_{\theta\le T}\mathsf{P}\Big(\sup_{t\le\triangle}|x^\varepsilon_{\theta+t}-x^\varepsilon_\theta|>\eta,\ \varTheta_C=\infty\Big)=-\infty.$$

An entry of X^ε_t satisfies

$$x^\varepsilon_t=x^\varepsilon_0+\int_0^t\gamma^\varepsilon_s ds+\varepsilon m^\varepsilon_t,$$

where γ^ε_t is $\mathbf{F}^\varepsilon$-adapted continuous random process and m_t is $\mathbf{F}^\varepsilon$-continuous martingale with $\langle m^\varepsilon\rangle_t=\int_0^t\mu^\varepsilon_s ds$. Since b and σ are locally Lipschitz continuous functions, there is a constant l_C, such that $|\gamma^\varepsilon_{\varTheta_C\wedge t}|\le l_C$ and $\mu^\varepsilon_{\varTheta_C\wedge t}\le l_C$. Taking into account that

$$\Big\{\sup_{t\le\triangle}\Big|\int_\theta^{\theta+t}\gamma^\varepsilon_s ds\Big|\ge\eta,\ \varTheta_C=\infty\Big\}\subseteq\big\{l_C\triangle\ge\eta\big\}=\varnothing,\ \text{for }\triangle<\eta/l_C,$$

it is left to verify

$$\lim_{\triangle\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2\log\sup_{\theta\le T}\mathsf{P}\Big(\sup_{t\le\triangle}|\varepsilon m^\varepsilon_{\theta+t}-\varepsilon m^\varepsilon_\theta|>\eta,\ \varTheta_C=\infty\Big)=-\infty.$$

Due to the obvious inclusion

$$\begin{aligned}\Big\{\sup_{t\le\triangle}|\varepsilon m^\varepsilon_{\theta+t}-\varepsilon m^\varepsilon_\theta|>\eta,\ \varTheta_C=\infty\Big\}&=\\ &\Big\{\sup_{t\le\triangle}|\varepsilon m^\varepsilon_{\varTheta_C\wedge(\theta+t)}-\varepsilon m^\varepsilon_{\varTheta_C\wedge\theta}|>\eta,\ \varTheta_C=\infty\Big\}\\ &\subseteq\Big\{\sup_{t\le\triangle}|\varepsilon m^\varepsilon_{\varTheta_C\wedge(\theta+t)}-\varepsilon m^\varepsilon_{\varTheta_C\wedge\theta}|>\eta\Big\},\end{aligned}$$

we shall verify

$$\overline{\lim_{\triangle\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2\log\sup_{\theta\le T}\mathsf{P}\Big(\sup_{t\le\triangle}|\varepsilon m^\varepsilon_{\varTheta_C\wedge(\theta+t)}-\varepsilon m^\varepsilon_{\varTheta_C\wedge\theta}|>\eta\Big)=-\infty.$$

Notice that $n^\varepsilon_t:=\varepsilon m^\varepsilon_{\varTheta_C\wedge(\theta+t)}-\varepsilon m^\varepsilon_{\varTheta_C\wedge\theta}$ is a continuous martingale relative to $(\mathscr{F}^\varepsilon_{\varTheta_C\wedge\theta+t})_{t\ge 0}$ (see e.g. Ch. 4, §7 in [13]) with $\langle n^\varepsilon\rangle_t=\varepsilon^2\int_{\varTheta_C\wedge\theta}^{\varTheta_C\wedge(\theta+t)}\mu^\varepsilon_s ds\le\varepsilon^2 l_C t$. By the statement (d) of Proposition A.1, $\mathsf{P}\big(\sup_{t\le\triangle}|n^\varepsilon_t|\ge\eta\big)\le 2e^{-\eta^2/(2l_C\varepsilon^2\triangle)}$, so that $\overline{\lim}_{\varepsilon\to 0}\,\varepsilon^2\log\mathsf{P}\big(\sup_{t\le\triangle}|n^\varepsilon_t|\ge\eta\big)\le-\frac{\eta^2}{2l_C\triangle}\xrightarrow[\triangle\to 0]{}-\infty$. □

5. **Local LDP upper bound**

We start with the observation that (3.3) holds if for any $T>0$

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le T}\|X^\varepsilon_t-u_t\|\le\delta,\ \varTheta_C=\infty\Big)\le -J_T(u),\tag{5.1}$$

since by the inclusion

$$\Big\{\sup_{t\le T}\|X^\varepsilon_t-u_t\|\le\delta\Big\}\subseteq\Big\{\sup_{t\le T}\|X^\varepsilon_t-u_t\|\le\delta,\ \varTheta_C=\infty\Big\}\bigcup\Big\{\varTheta_C\le T\Big\}$$

we have

$$\varlimsup_{\delta\to 0}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log \mathsf{P}\Big(\sup_{t\le T}\|X^\varepsilon_t-u_t\|\le\delta\Big)$$
$$\le \varlimsup_{\delta\to 0}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log \mathsf{P}\Big(\sup_{t\le T}\|X^\varepsilon_t-u_t\|\le\delta,\ \Theta_C=\infty\Big)\bigvee\varlimsup_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}(\Theta_C\le T),$$

and, by (4.1), the last term goes to $-\infty$ as $C\to\infty$.

The proof for $u_0\ne x_0$ or $du_t \not\ll dt$ is standard (see e.g. [4]) and is omitted. The rest of the proof is split into several steps.

5.1. $\boldsymbol{u_0=x_0,\ du_t\ll dt,\ \int_0^T\|\dot u_s\|^2ds<\infty}$**.** Define the set

$$\mathfrak{A}=\Big\{\sup_{t\le T}\|X^\varepsilon_t-u_t\|\le\delta,\ \Theta_C=\infty\Big\}.$$

With a continuously differentiable vector-valued function $\lambda(s)$ of dimension d, let us introduce a continuous local martingale $U_t=\int_0^t(\lambda(s),\varepsilon\sigma(X^\varepsilon_s)dB_s)$ and its martingale exponential $\mathfrak{z}_t=e^{U_t-0.5\langle U\rangle_t}$, where

$$\langle U\rangle_t=\int_0^t\varepsilon^2(\lambda(s),a(X^\varepsilon_s)\lambda(s))ds.$$

It is well known that $\mathfrak{z}_t$ is a continuous positive local martingale, as well as a supermartingale. Consequently, $\mathsf{E}\mathfrak{z}_T\le 1$ and, therefore,

$$1\ge \mathsf{E}I_{\{\mathfrak{A}\}}\mathfrak{z}_T. \tag{5.2}$$

The required upper bound for $\mathsf{P}(\mathfrak{A})$ is obtained by estimating $\mathfrak{z}_T$ from below on $\mathfrak{A}$. Since $U_t=\int_0^t(\lambda(s),dX^\varepsilon_s-b(X^\varepsilon_s)ds)$,

$$\begin{aligned}
&U_T-0.5\langle U\rangle_T=\\
&\int_0^T\Big[(\lambda(s),dX^\varepsilon_s-b(X^\varepsilon_s)ds)-\frac{\varepsilon^2}{2}(\lambda(s),a(X^\varepsilon_s)\lambda(s))ds\Big]=\\
&\int_0^T\Big[(\lambda(s),\dot u_s-b(u_s))-\frac{\varepsilon^2}{2}(\lambda(s),a(u_s)\lambda(s))\Big]ds\\
&+\int_0^T(\lambda(s),dX^\varepsilon_s-\dot u_sds)+\int_0^T(\lambda(s),b(u_s)-b(X^\varepsilon_s)ds\\
&+\int_0^T\frac{\varepsilon^2}{2}(\lambda(s),[a(u_s)-a(X^\varepsilon_s)]\lambda(s))ds.
\end{aligned}\tag{5.3}$$

We derive lower bounds on the set $\mathfrak{A}$ for each term in the right hand side of (5.3). Applying the Itô formula to $(\lambda(t),X^\varepsilon_t-u_t)$, and taking into account that $X^\varepsilon_0=u_0$, we find that

$$(\lambda(T),X^\varepsilon_T-u_T)=\int_0^T(\lambda(s),dX^\varepsilon_s-\dot u_sds)+\int_0^T(\dot\lambda(s),X^\varepsilon_s-u_s)ds.$$

Therefore,

$$\int_0^T(\lambda(s),dX^\varepsilon_s-\dot u_sds)$$
$$\ge-\Big|(\lambda(T),X^\varepsilon_T-u_T)\Big|-\Big|\int_0^T(\dot\lambda(s),X^\varepsilon_s-u_s)ds\Big|\ge -r_1\delta,$$

with $r_1 := r_1(\lambda, T, C) \geq 0$, independent of ε.

Further, with $r_i := r_i(\lambda, T, C) \geq 0$, $i = 2, 3$, due to the local Lipschitz continuity of σ and a, we find that

$$\int_0^T (\lambda(s), b(u_s) - b(X_s^\varepsilon))ds \geq -r_2(\lambda, C, T)\delta$$

$$\int_0^T \frac{\varepsilon^2}{2}(\lambda(s), [a(u_s) - a(X_s^\varepsilon)]\lambda(s))ds \geq -\varepsilon^2 r_3(\lambda, C, T)\delta.$$

Hence with $r := r_1 + r_2 + \varepsilon^2 r_3$,

$$\log \mathfrak{z}_T \geq \int_0^T \Big[(\lambda(s), \dot{u}_s - b(u_s)) - \frac{\varepsilon^2}{2}(\lambda(s), a(u_s)\lambda(s))\Big]ds - r(\lambda, T, C)\delta.$$

Set $\nu(s) = \varepsilon^2\lambda(s)$ and rewrite the above inequality as:

$$\log \mathfrak{z}_T \geq \frac{1}{\varepsilon^2}\int_0^T \Big[(\nu(s), \dot{u}_s - b(u_s)) - \frac{1}{2}(\nu(s), a(u_s)\nu(s))\Big]ds - r\Big(\frac{\nu}{\varepsilon^2}, T, C\Big)\delta.$$

This lower bound, along with (5.2), provides the following upper bound

$$\varepsilon^2 \log \mathsf{P}\big(\mathfrak{A}\big) \leq -\int_0^T \Big[(\nu(s), \dot{u}_s - b(u_s)) - \frac{1}{2}(\nu(s), a(u_s)\nu(s))\Big]ds + \varepsilon^2 r\Big(\frac{\nu}{\varepsilon^2}, T, C\Big)\delta.$$

Clearly $\overline{\lim}_{\varepsilon\to 0}\, \varepsilon^2 r\big(\frac{\nu}{\varepsilon^2}, T, C\big) < \infty$ and, hence,

$$\overline{\lim_{\delta\to 0}}\, \overline{\lim_{\varepsilon\to 0}}\, \varepsilon^2 \log \mathsf{P}\big(\mathfrak{A}\big) \leq -\int_0^T \Big[(\nu(s), \dot{u}_s - b(u_s)) - \frac{1}{2}(\nu(s), a(u_s)\nu(s))\Big]ds. \quad (5.4)$$

Since the left hand side of (5.4) is independent of $\nu(s)$, (5.1) is derived by minimizing the right hand side of (5.4) with respect to $\nu(s)$. Two difficulties arise on the way to direct minimization:

- the matrix $a(u_s)$ may be singular
- the entries of $\nu(s)$ should be continuously differentiable functions.

Assume first $a(u_s)$ is a positive definite matrix, uniformly in s, and write

$$(\nu(s), \dot{u}_s - b(u_s)) - \frac{1}{2}(\nu(s), a(u_s)\nu(s)) = \frac{1}{2}\|\dot{u}_s - b(u_s)\|^2_{a^{-1}(u_s)} - \frac{1}{2}\Big\|a^{1/2}(u_s)\big(\nu(s) - a^{-1}(u_s)[\dot{u}_s - b(u_s)]\big)\Big\|^2.$$

If the entries of $a^{-1}(u_s)[\dot{u}_s - b(u_s)]$ are continuously differentiable functions, then, by taking $\nu(s) \equiv -a^{-1}(u_s)[\dot{u}_s - b(u_s)]$ we find that

$$\overline{\lim_{\delta\to 0}}\, \overline{\lim_{\varepsilon\to 0}}\, \varepsilon^2 \log \mathsf{P}\big(\mathfrak{A}\big) \leq -\frac{1}{2}\int_0^T \|\dot{u}_s - b(u_s)\|^2_{a^{-1}(u_s)}ds. \quad (5.5)$$

In the general case, due to $\int_0^T \|\dot{u}_s\|^2 ds < \infty$, the entries of $a^{-1}(u_s)[\dot{u}_s - b(u_s)]$ are square integrable with respect to the Lebesgue measure on $[0, T]$. Choose a maximizing sequence $\nu_n(s)$, $n \geq 1$, of continuously differentiable functions such that $\lim_{n\to\infty}\int_0^T \big\|\nu_n(s) - a^{-1}(u_s)[\dot{u}_s - b(u_s)]\big\|^2 ds = 0$. Since all the entries of $a(u_s)$ are uniformly bounded on $[0,T]$

$$\lim_{n\to\infty}\int_0^T \big\|a^{1/2}(u_s)\big(\nu_n(s) - a^{-1}(u_s)[\dot{u}_s - b(u_s)]\big)\big\|^2 ds = 0$$

and (5.5) holds too.

Now we drop the uniform nonsingularity assumption of $a(u_s)$. The upper bound in (5.5) remains valid with $a(u_s)$ replaced by $a_\beta(u_s) \equiv a(u_s) + \beta\mathbf{I}$, where β is a positive number and $\mathbf{I}$ is $(d\times d)$-unit matrix:

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2 \log \mathsf{P}\big(\mathfrak{A}\big) \leq -\frac{1}{2}\int_0^T \|\dot{u}_s - b(u_s)\|^2_{[a(u_s)+\beta\mathbf{I}]^{-1}} ds.$$

For any fixed s, the function $\|\dot{u}_s - b(u_s)\|^2_{[a(u_s)+\beta\mathbf{I}]^{-1}}$ increases with $\beta\downarrow 0$ and by Lemma B.1 possesses the limit

$$\lim_{\beta\to 0}\|\dot{u}_s - b(u_s)\|^2_{[a(u_s)+\beta\mathbf{I}]^{-1}} = \begin{cases} \|\dot{u}_s - b(u_s)\|^2_{a^\oplus(u_s)}, & a(u_s)a^\oplus(u_s)[\dot{u}_s - b(u_s)] = [\dot{u}_s - b(u_s)] \\ \infty, & \text{otherwise.}\end{cases}$$

Thus the required upper bound

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2 \log \mathsf{P}\big(\mathfrak{A}\big) \leq \begin{cases} -\int_0^T \frac{1}{2}\|\dot{u}_s - b(u_s)\|^2_{a^\oplus(u_s)} ds, & a(u_s)a^\oplus(u_s)[\dot{u}_s - b(u_s)] = [\dot{u}_s - b(u_s)], \text{ a.s.} \\ \infty, & \text{otherwise}\end{cases}$$

follows by the monotone convergence theorem.

5.2. $\boldsymbol{u_0 = x_0,\ du_t \ll dt,\ \int_0^T \|\dot{u}_s\|^2 ds = \infty.}$ We emphasize that $du_t \ll dt$ on $[0,T]$ implies $\int_0^T \|\dot{u}_s\| ds < \infty$ and return to the upper bound from (5.4). Since b and σ are locally Lipschitz, one can choose a constant L (depending on $u(s)$), so that, $|(\nu(s), b(u_s))| \leq \|b(u_s)\|\|\nu(s)\| \leq L\|\nu(s)\|$ and $(\nu(s), a(u_s)\nu(s)) \leq L\|\nu(s)\|^2$. Then, (5.4) implies

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2 \log \mathsf{P}\big(\mathfrak{A}\big) \leq -\int_0^T \Big[(\nu(s),\dot{u}_s) - L\|\nu(s)\| - \frac{L}{2}\|\nu(s)\|^2\Big] ds.$$

Let $\nu_n(s)$ be a sequence of continuously differentiable functions, approximating the bounded (for each fixed $p>0$) function $L^{-1}\dot{u}_s I_{\{\|\dot{u}_s\|\leq p\}}$ in the sense that $\lim_{n\to\infty}\int_0^T \|\frac{1}{L}\dot{u}_s I_{\{\|\dot{u}_s\|\leq p\}} - \nu_n(s)\|^2 ds = 0$. Thus,

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2 \log \mathsf{P}\big(\mathfrak{A}\big) \leq -\frac{1}{2L}\underbrace{\int_0^T \big\|\dot{u}_s\big\|^2 I_{\{\|\dot{u}_s\|\leq p\}} ds}_{\uparrow\infty \text{ as } p\uparrow\infty} + \underbrace{\int_0^T \big\|\dot{u}_s\big\| ds}_{<\infty} \xrightarrow[p\to\infty]{} -\infty$$

□

6. **Local LDP lower bound.**

If $\overline{\lim}_{\delta\to 0}\overline{\lim}_{\varepsilon\to 0}\varepsilon^2\log \mathsf{P}\big(\sup_{t\le \Theta_C\wedge T}|X^\varepsilon_t-u_t|\le\delta\big)\le -J_T(u)=-\infty$, then the corresponding local LDP lower bound is $-\infty$ as well and hence only the case $J_T(u)<\infty$ is to be considered, i.e. we may restrict ourselves to analyzing test functions with the properties:

$$\begin{aligned}
&\textbf{(i)}\quad u_0=x_0\\
&\textbf{(ii)}\quad du_t\ll dt\\
&\textbf{(iii)}\quad a(u_t)a^{\oplus}(u_t)[\dot u_t-b(u_t)]=[\dot u_t-b(u_t)]\quad \text{a.s.}\\
&\textbf{(iv)}\quad \int_0^T\|\dot u_t-b(u_t)\|^2_{a^{\oplus}(u_t)}dt<\infty,\ \forall\ T>0\\
&\textbf{(v)}\quad \int_0^T\|\dot u_t\|^2dt<\infty.
\end{aligned}\tag{6.1}$$

Another helpful observation is that (3.4) holds if for any $C>0$

$$\varliminf_{\delta\to 0}\varliminf_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le\Theta_C\wedge T}\|X^\varepsilon_t-u_t\|\le\delta\Big)\ge -J_T(u)\tag{6.2}$$

due to

$$\Big\{\sup_{t\le\Theta_C\wedge T}\|X^\varepsilon_t-u_t\|\le\delta\Big\}\subseteq\Big\{\sup_{t\le T}\|X^\varepsilon_t-u_t\|\le\delta\Big\}\bigcup\Big\{\Theta_C\le T\Big\}$$

and (4.1).

6.1. **Nonsingular $a(x)$.** In this section, the matrix $a(x)$ is assumed to be uniformly nonsingular in $x\in\mathbb{R}$, in the sense that $a(x)\ge\beta\mathbf{I}$ for a positive number β. Let $\lambda(s):=\sigma^{-1}(X^\varepsilon_s)\big[\dot u_s-b(X^\varepsilon_s)\big]$ and introduce a martingale $U_t=\int_0^{\Theta_C\wedge t}\frac{1}{\varepsilon}(\lambda(s),dB_s)$ and its martingale exponential $\mathfrak{z}_t=e^{U_t-0.5\langle U\rangle_t}$, $t\le T$, where $\langle U\rangle_t=\int_0^{\Theta_C\wedge t}\frac{1}{\varepsilon^2}\|\lambda(s)\|^2ds$.

By (**iv**) and (**v**) of (6.1), $\langle U\rangle_T\le$ const. and hence $\mathsf{E}\mathfrak{z}_T=1$. We use this fact in order to define a new probability measure Q^ε by $d\mathsf{Q}^\varepsilon=\mathfrak{z}_Td\mathsf{P}$. Since $\mathfrak{z}_T$ is positive P-a.s., $\mathsf{P}\ll\mathsf{Q}^\varepsilon$ as well and $d\mathsf{P}=\mathfrak{z}_T^{-1}d\mathsf{Q}^\varepsilon$.

We proceed with the proof of (6.2) by applying

$$\mathsf{P}(\widetilde{\mathfrak{A}})=\int_{\widetilde{\mathfrak{A}}}\mathfrak{z}_T^{-1}d\mathsf{Q}^\varepsilon\tag{6.3}$$

to the set $\widetilde{\mathfrak{A}}=\big\{\sup_{t\le\Theta_C\wedge T}\|X^\varepsilon_t-u_t\|\le\delta\big\}$, and estimating from below the right hand side in (6.3). In order to realize this program, it is convenient to have a semimartingale description of the process $X^\varepsilon_{\Theta_C\wedge t}$ under Q^ε. Recall that the random process $B_{\Theta_C\wedge t}$ is a martingale under P with the variation process $\langle B\rangle_{\Theta_C\wedge t}\equiv(\Theta_C\wedge t)\mathbf{I}$. It is well known (see e.g. Theorem 2, Ch. 4,§5 in [13]) that $B_{\Theta_C\wedge t}$ is a continuous semimartingale under Q^ε with the decomposition $B_{\Theta_C\wedge t}=\widetilde B_t+A^B_t$, where $\widetilde B_t$ is a martingale (under Q^ε) with $\langle\widetilde B\rangle_t\equiv\langle B\rangle_{\Theta_C\wedge t}$ and, by the Girsanov theorem,

$$A^B_t=\int_0^{\Theta_C\wedge t}\frac{1}{\varepsilon}\sigma^{-1}(X^\varepsilon_s)[\dot u_s-b(X^\varepsilon_s)]ds.$$

In particular,

$$X^\varepsilon_{\Theta_C\wedge t}=u_{\Theta_C\wedge t}+\varepsilon\int_0^{\Theta_C\wedge t}\sigma(X^\varepsilon_s)d\widetilde{B}_s,\ t\le T,\quad \mathsf{Q}^\varepsilon\text{-a.s.}$$

As the next preparatory step we derive the semimartingale decomposition of U_t under Q^ε. As before, the continuous martingale U_t under P is transformed to a semimartingale under Q^ε:

$$U_t=\widetilde{U}_t+A^U_t$$

with continuous Q^ε-martingale $\widetilde{U}_t$, having the variation process $\langle\widetilde{U}\rangle_t\equiv\langle U\rangle_t$, P- and Q^ε-a.s., and a continuous drift $A^U_t\equiv\langle U\rangle_t$.

Thus, $U_t=\widetilde{U}_t+\langle U\rangle_t,\ t\le T,$ Q^ε-a.s. and, thereby, $\mathfrak{z}^{-1}_T=e^{-\widetilde{U}_T-\frac12\langle U\rangle_T}$. Consequently, (6.3) is transformed to

$$\begin{aligned}\mathsf{P}(\widetilde{\mathfrak{A}})&=\int_{\widetilde{\mathfrak{A}}}\exp\Big(-\widetilde{U}_T-\frac12\langle U\rangle_T\Big)d\mathsf{Q}^\varepsilon\\&=\int_{\widetilde{\mathfrak{A}}}\exp\Big(-\widetilde{U}_T-\frac{1}{2\varepsilon^2}\int_0^{\Theta_C\wedge T}\|\dot u_s-b(X^\varepsilon_s)\|^2_{a^{-1}(X^\varepsilon_s)}ds\Big)d\mathsf{Q}^\varepsilon.\end{aligned}$$

We are now in the position to derive a lower bound for the right hand side. Replacing $\widetilde{\mathfrak{A}}$ with a smaller set $\widetilde{\mathfrak{A}}\cap\mathfrak{B}$, where $\mathfrak{B}=\big\{\big|\varepsilon^2\widetilde{U}_T\big|\le\eta\big\}$, write

$$\mathsf{P}(\widetilde{\mathfrak{A}})\ge\int_{\widetilde{\mathfrak{A}}\cap\mathfrak{B}}\exp\Big(-\frac{\eta}{\varepsilon^2}-\frac{1}{2\varepsilon^2}\int_0^{\Theta_C\wedge T}\|\dot u_s-b(X^\varepsilon_s)\|^2_{a^{-1}(X^\varepsilon_s)}ds\Big)d\mathsf{Q}^\varepsilon.$$

By the local Lipschitz continuity of b,σ and the uniform nonsingularity of $a(x)$,

$$\Big|\|\dot u_s-b(X^\varepsilon_s)\|^2_{a^{-1}(X^\varepsilon_s)}-\|\dot u_s-b(u_s)\|^2_{a^{-1}(u_s)}\Big|\le l_C(\|\dot u_s\|+1)^2\delta,\ \delta\le1,$$

on the set $\widetilde{\mathfrak{A}}\cap\mathfrak{B}$ for any $s\le\Theta_C\wedge T$. Then,

$$\begin{aligned}\mathsf{P}(\widetilde{\mathfrak{A}})&\ge\int_{\widetilde{\mathfrak{A}}\cap\mathfrak{B}}\exp\Big(-\frac{\eta}{\varepsilon^2}-\frac{\delta l_C}{\varepsilon^2}\int_0^T(\|\dot u_s\|+1)^2ds\\&\qquad-\frac{1}{2\varepsilon^2}\int_0^{\Theta_C\wedge T}\|\dot u_s-b(u_s)\|^2_{a^{-1}(u_s)}ds\Big)d\mathsf{Q}^\varepsilon\\&\ge\int_{\widetilde{\mathfrak{A}}\cap\mathfrak{B}}\exp\Big(-\frac{\eta}{\varepsilon^2}-\frac{\delta l_C}{\varepsilon^2}\int_0^T(\|\dot u_s\|+1)^2ds\\&\qquad-\frac{1}{2\varepsilon^2}\int_0^{T}\|\dot u_s-b(u_s)\|^2_{a^{-1}(u_s)}ds\Big)d\mathsf{Q}^\varepsilon.\end{aligned}$$

Consequently,

$$\varliminf_{\varepsilon\to0}\varepsilon^2\log\mathsf{P}(\widetilde{\mathfrak{A}})\ge-\eta-\delta l_C\int_0^T(\|\dot u_s\|+1)^2ds-J_T(u)+\varliminf_{\varepsilon\to0}\varepsilon^2\log\mathsf{Q}^\varepsilon\big(\widetilde{\mathfrak{A}}\cap\mathfrak{B}\big).$$

We prove now that $\varliminf_{\varepsilon\to0}\varepsilon^2\log\mathsf{Q}^\varepsilon\big(\widetilde{\mathfrak{A}}\cap\mathfrak{B}\big)=0$ by showing

$$\lim_{\varepsilon\to0}\mathsf{Q}^\varepsilon\big(\varOmega\setminus\widetilde{\mathfrak{A}}\big)=0\quad\text{and}\quad\lim_{\varepsilon\to0}\mathsf{Q}^\varepsilon\big(\varOmega\setminus\mathfrak{B}\big)=0.$$

To this end, recall that

$$\begin{aligned}\Omega\setminus\widetilde{\mathfrak{A}} &= \Big\{\varepsilon\sup_{t\le T}\Big\|\int_0^{\Theta_C\wedge t}\sigma(X_s^\varepsilon)d\widetilde{B}_s\Big\|>\delta\Big\}\\ \Omega\setminus\mathfrak{B} &= \Big\{\varepsilon\Big\|\int_0^{\Theta_C\wedge T}\sigma^{-1}(X_s^\varepsilon)[\dot u_s-b(X_s^\varepsilon)]d\widetilde{B}_s\Big\|>\eta\Big\}.\end{aligned}\tag{6.4}$$

We verify (6.4) componentwise. Let L_t^ε denote any entry of $\int_0^{\Theta_C\wedge t}\sigma(X_s^\varepsilon)d\widetilde{B}_s$ or $\int_0^{\Theta_C\wedge t}\sigma^{-1}(X_s^\varepsilon)[\dot u_s-b(X_s^\varepsilon)]d\widetilde{B}_s$. We show that

$$\lim_{\varepsilon\to0}\mathsf{Q}^\varepsilon\big(\varepsilon\sup_{t\le T}\big|L_t^\varepsilon\big|>\delta\big)=0 \text{ and } \lim_{\varepsilon\to0}\mathsf{Q}^\varepsilon\big(\varepsilon\big|L_T^\varepsilon\big|>\delta\big)=0.\tag{6.5}$$

In both cases, L_t^ε is a continuous Q^ε-martingale with $\langle L^\varepsilon\rangle_t=\int_0^t g(s)ds$ and $\int_\Omega\int_0^T g(s)dsd\mathsf{Q}^\varepsilon<\infty$. Then (6.5) holds by Doob's inequality:

$$\lim_{\varepsilon\to0}\mathsf{Q}^\varepsilon\big(\varepsilon\sup_{t\le T}\big|L_t^\varepsilon\big|>\delta\big)\le\frac{4\varepsilon^2}{\delta^2}\int_\Omega\int_0^T g(s)dsd\mathsf{Q}^\varepsilon\xrightarrow[\varepsilon\to0]{}0.$$

Now, for any fixed δ and η,

$$\varliminf_{\varepsilon\to0}\varepsilon^2\log\mathsf{P}(\widetilde{\mathfrak{A}})\ge-\eta-\delta l_C\int_0^T(\|\dot u_s\|+1)^2ds-J_T(u).$$

The required lower bound

$$\varliminf_{\delta\to0}\varliminf_{\varepsilon\to0}\varepsilon^2\log\mathsf{P}(\widetilde{\mathfrak{A}})\ge-J_T(u)$$

follows by taking $\lim_{\eta\to0}\lim_{\delta\to0}$. □

6.2. **General $a(x)$.** This part of the proof requires perturbation arguments. The idea is to use the already obtained local LDP lower bound for the uniformly nonsingular $a(x)$. Let W_t be a standard d dimensional Brownian motion, independent of B_t, defined on the same stochastic basis. Since b and σ are assumed to be locally Lipschitz continuous, one can introduce the perturbed diffusion process controlled by a free parameter $\beta\in(0,1]$:

$$X_t^{\varepsilon,\beta}=x_0+\int_0^t b(X_s^{\varepsilon,\beta})ds+\varepsilon\int_0^t[\sigma(X_s^{\varepsilon,\beta})dB_s+\sqrt{\beta}dW_s].\tag{6.6}$$

The process $X_t^{\varepsilon,\beta}$, defined in (6.6), solves the Itô equation $X_t^{\varepsilon,\beta}=x_0+\int_0^t b(X_s^{\varepsilon,\beta})ds+\varepsilon\int_0^t[a(X_s^{\varepsilon,\beta})+\beta\mathbf{I}]^{1/2}dB_s^\beta$ with respect to a standard Brownian motion $B_t^\beta=\int_0^t[a(X_s^{\varepsilon,\beta})+\beta\mathbf{I}]^{-1/2}[\sigma(X_s^{\varepsilon,\beta})dB_s+\sqrt{\beta}dW_s]$. Then the family $\{(X_t^{\varepsilon,\beta})_{t\le T}\}_{\varepsilon\to0}$ satisfies the local LDP lower bound. Indeed, the matrix $a_\beta(x)$ is uniformly nonsingular, its entries are locally bounded and satisfy the assumption (H-3) of Theorem 2.1 since

$$\frac{(x,a_\beta(x)x)}{\|x\|\ |(x,b(x))|}=\frac{(x,a(x)x)}{\|x\|\ |(x,b(x))|}+\beta\frac{\|x\|}{|(x,b(x))|}$$

and $\frac{\|x\|}{|(x,b(x))|}$ converges to zero as $\|x\|\to\infty$ by (H-2). In particular, with $\Theta_C^\beta=\inf\{t:\|X_t^{\varepsilon,\beta}\|\ge C\}$ and $u_0=x_0$, $du_t\ll dt$, $\int_0^T\|\dot u_t\|^2dt<\infty$, we have

$$\varliminf_{\delta\to 0}\varliminf_{\varepsilon\to 0}\varepsilon^2\log \mathsf{P}\Big(\sup_{t\le \Theta^\beta_C\wedge T}\|X^{\varepsilon,\beta}_t-u_t\|\le\delta\Big)\ge -\frac12\int_0^T\|\dot u_s-b(u_s)\|^2_{(a(u_s)+\beta\mathbf{I})^{-1}}ds. \quad (6.7)$$

Further, we will use (6.7) to establish

$$\varliminf_{\delta\to 0}\varliminf_{\varepsilon\to 0}\varepsilon^2\log \mathsf{P}\Big(\sup_{t\le T}\|X^{\varepsilon}_t-u_t\|\le\delta\Big)\ge -\frac12\int_0^T\|\dot u_s-b(u_s)\|^2_{a^{\oplus}(u_s)}ds. \quad (6.8)$$

To this end, we introduce the filtration $\mathbf{G}^\varepsilon=(\mathscr{G}^\varepsilon_t)_{t\ge 0}$, with the general conditions, generated by $(X^\varepsilon_t, X^{\varepsilon,\beta}_t)_{t\ge 0}$ and notice that both Θ_C (see (4.2)) and Θ^β_C are stopping times relative to $\mathbf{G}^\varepsilon$. Hence,

$$\tau^\beta_C=\Theta_C\wedge\Theta^\beta_C \quad (6.9)$$

is a stopping time as well relative to $\mathbf{G}^\varepsilon$. Obviously,

$$\lim_{C\to\infty}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\big(\tau^\beta_C\le T\big)=-\infty.$$

However, the proof of (6.8) requires a stronger property:

$$\lim_{C\to\infty}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log\sup_{\beta\in(0,1]}\mathsf{P}\Big(\tau^\beta_C\le T\Big)=-\infty. \quad (6.10)$$

It is clear, that (6.10) is valid if it is valid with τ^β_C replaced by Θ^β_C. The latter is verified along the lines of Lemma 4.1 proof:

$$\begin{aligned}
&\varepsilon^2\log\sup_{\beta\in(0,1]}\mathsf{P}\big(\Theta^\beta_C\le T\big)\le -\inf_{\|x\|\ge C}V(x)+V(x_0)\\
&+\frac{T\varepsilon^2}{2}\sup_{\beta\in(0,1]}\sup_{\|x\|\le C}\Big|\operatorname{trace}\big(\Psi(x)[a(x)+\beta\mathbf{I}]\big)\Big|+T\sup_{\beta\in(0,1]}\sup_{\|x\|\le L}\big|\mathfrak{D}_\beta V(x)\big|\\
&\quad\xrightarrow[\varepsilon\to 0]{}-\inf_{\|x\|\ge C}V(x)+V(x_0)+T\sup_{\beta\in(0,1]}\sup_{\|x\|\le L}\big|\mathfrak{D}_\beta V(x)\big|\xrightarrow[C\to\infty]{}-\infty,
\end{aligned}$$

where $\mathfrak{D}_\beta V(x) = (\nabla V(x), b(x)) + \frac{1}{2}(\nabla V(x), a_\beta(x)\nabla V(x))$. We are now in the position to prove (6.8). With $\delta \le \beta^{1/4}$, write

$$
\begin{aligned}
&\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon,\beta}-u_t\|\le\delta\Big\}\\
&=\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon,\beta}-u_t\|\le\delta\Big\}\bigcap\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-X_t^{\varepsilon,\beta}\|\le\beta^{1/4}\Big\}\\
&\quad\bigcup\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon,\beta}-u_t\|\le\delta\Big\}\bigcap\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-X_t^{\varepsilon,\beta}\|>\beta^{1/4}\Big\}\\
&\subseteq\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon,\beta}-u_t\|\le\beta^{1/4}\Big\}\bigcap\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-X_t^{\varepsilon,\beta}\|\le\beta^{1/4}\Big\}\\
&\quad\bigcup\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-X_t^{\varepsilon,\beta}\|>\beta^{1/4}\Big\}\\
&\subseteq\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-u_t\|\le2\beta^{1/4}\Big\}\bigcup\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-X_t^{\varepsilon,\beta}\|>\beta^{1/4}\Big\}\\
&\subseteq\Big\{\sup_{t\le T}\|X_t^{\varepsilon}-u_t\|\le2\beta^{1/4}\Big\}\bigcup\Big\{\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-X_t^{\varepsilon,\beta}\|>\beta^{1/4}\Big\}\\
&\qquad\qquad\bigcup\Big\{\tau_C^\beta\le T\Big\}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathsf{P}\Big(\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon,\beta}-u_t\|\le\delta\Big)\le3\Big\{&\mathsf{P}\Big(\sup_{t\le T}\|X_t^{\varepsilon}-u_t\|\le2\beta^{1/4}\Big)\\
&\bigvee\mathsf{P}\Big(\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-X_t^{\varepsilon,\beta}\|>\beta^{1/4}\Big)\bigvee\mathsf{P}\big(\tau_C^\beta\le T\big)\Big\}.
\end{aligned}
$$

Clearly, Θ_C^β can be replaced by τ_C^β, and so

$$
\begin{aligned}
-\frac{1}{2}\int_0^T\|\dot u_s-b(u_s)\|^2_{(a(u_s)+\beta\mathbf{I})^{-1}}ds\le&\varliminf_{\varepsilon\to0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le T}\|X_t^{\varepsilon}-u_t\|\le2\beta^{1/4}\Big)\\
&\bigvee\varlimsup_{\varepsilon\to0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-X_t^{\varepsilon,\beta}\|>\beta^{1/4}\Big)\\
&\bigvee\varlimsup_{\varepsilon\to0}\varepsilon^2\log\sup_{\beta\in(0,1]}\mathsf{P}\Big(\tau_C^\beta\le T\Big).
\end{aligned}
\tag{6.11}
$$

Next we use the following facts:

(1) by Lemma B.1 and (6.1),

$$\lim_{\beta\to0}\int_0^T\|\dot u_s-b(u_s)\|^2_{(a(u_s)+\beta\mathbf{I})^{-1}}ds=\int_0^T\|\dot u_s-b(u_s)\|^2_{a^\oplus(u_s)}ds;$$

(2) by Lemma C.1,

$$\lim_{\beta\to0}\varlimsup_{\varepsilon\to0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le\tau_C^\beta\wedge T}\|X_t^{\varepsilon}-X_t^{\varepsilon,\beta}\|>\beta^{1/4}\Big)=-\infty;$$

(3) by (6.10), $\lim_{C\to\infty}\overline{\lim}_{\varepsilon\to 0}\varepsilon^2\log\sup_{\beta\in(0,1]}\mathsf{P}\Big(\tau_C^\beta\le T\Big)=-\infty$.

Hence, passing to the limit $\beta\to 0$ and then $C\to\infty$ in (6.11) and taking into account (1)-(3), one gets the required lower bound

$$\underline{\lim}_{\beta\to 0}\,\underline{\lim}_{\varepsilon\to 0}\,\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le T}\|X_t^\varepsilon-u_t\|\le 2\beta^{1/4}\Big)\ge-\frac{1}{2}\int_0^T\|\dot u_s-b(u_s)\|^2_{a^\oplus(u_s)}ds.$$

□

Appendix A. Exponential estimates for martingales

Proposition A.1. (Lemma A.1 in [10]) *Let $M=(M_t)_{t\ge 0}$, $M_t\in\mathbb{R}$, be a continuous local martingale with $M_0=0$ and the predictable variation process $\langle M\rangle_t$ defined on some stochastic basis with general conditions. Let τ be a stopping time, α and B positive constants and $\mathfrak{A}$ some measurable set.*

(a) *if $M_\tau-\frac{1}{2}\langle M\rangle_\tau\ge\alpha$ on $\mathfrak{A}$, then $\mathsf{P}(\mathfrak{A})\le e^{-\alpha}$;*
(b) *if $M_\tau\ge\alpha$ and $\langle M\rangle_\tau\le B$ on $\mathfrak{A}$, then $\mathsf{P}(\mathfrak{A})\le e^{-\frac{\alpha^2}{2B}}$;*
(c) $\mathsf{P}(\sup_{t\le T}|M_t|\ge\alpha,\langle M\rangle_T\le B)\le 2e^{-\frac{\alpha^2}{2B}}$;
(d) $\mathsf{P}(\sup_{t\le T}|M_t|\ge\alpha)\le 2e^{-\frac{\alpha^2}{2B}}\bigvee\mathsf{P}(\langle M\rangle_T>B)$.

Appendix B. Pseudoinverse of nonnegative definite matrices

Let $A^\oplus$ be the Moore-Penrose pseudoinverse matrix of A (see [1]).

Lemma B.1. *For $d\times d$ nonnegative definite matrix A and $x\in\mathbb{R}^d$,*

$$\lim_{\beta\to 0}(x,(A+\beta\mathbf{I})^{-1}x)=\begin{cases}\|x\|^2_{A^\oplus}, & AA^\oplus x=x\\ \infty, & \textit{otherwise.}\end{cases}$$

Proof. Let S be an orthogonal matrix, $S^*S=\mathbf{I}$, such that $D:=S^*AS$ is a diagonal matrix. Then, due to $S^*(A+\beta\mathbf{I})S=D+\beta\mathbf{I}$, we have $S^*(A+\beta\mathbf{I})^{-1}S=(D+\beta\mathbf{I})^{-1}$ and $S(D+\beta\mathbf{I})^{-1}S^*=(A+\beta\mathbf{I})^{-1}$. Write ($y:=S^*x$)

$$\begin{aligned}(x,(A+\beta\mathbf{I})^{-1}x)&=(x,S(D+\beta\mathbf{I})^{-1}S^*x)=(S^*x,(D+\beta\mathbf{I})^{-1}S^*x)\\&=(y,(D+\beta\mathbf{I})^{-1}y)=(y,(D+\beta\mathbf{I})^{-1}DD^\oplus y)\\&\quad+(y,(D+\beta\mathbf{I})^{-1}(\mathbf{I}-DD^\oplus)y).\end{aligned}$$

Since $\lim_{\beta\to 0}(D+\beta\mathbf{I})^{-1}DD^\oplus=D^\oplus$, one gets

$$\lim_{\beta\to 0}(y,(D+\beta\mathbf{I})^{-1}DD^\oplus y)=\|y\|^2_{D^\oplus}=\|x\|^2_{A^\oplus}$$

while $\lim_{\beta\to 0}(y,(D+\beta\mathbf{I})^{-1}(\mathbf{I}-DD^\oplus)y)\ne\infty$ only if $(\mathbf{I}-DD^\oplus)y=0$. Since the latter condition is nothing but $(\mathbf{I}-AA^\oplus)x=0$, the desired statement holds. □

Appendix C. Exponential negligibility of $X_t^{\varepsilon,\beta} - X_t^\varepsilon$

We start with an auxiliary result.

Proposition C.1. *Let Y_t be a nonnegative continuous semimartingale defined on a stochastic basis (with general conditions):*

$$Y_t = \int_0^t h_1(s)Y_s ds + \varepsilon\int_0^t h_2(s)Y_s dM'_s + \varepsilon\sqrt{\beta}\int_0^t h_3(s)\sqrt{Y_s}dM''_s + \varepsilon^2\beta\int_0^t h_4(s)ds, \quad \text{(C.1)}$$

where $h_i(s), i = 1,\dots,4$, are bounded predictable processes and M'_t, M''_t are continuous martingales, $d\langle M'\rangle_t = m'(t)dt$, $d\langle M''\rangle_t = m''(t)dt$, $\langle M', M''\rangle_t \equiv 0$ with bounded $m'(t)$ and $m''(t)$. Assume that for any $T > 0$ and $\beta > 0$,

$$\lim_{L\to\infty}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log \mathsf{P}\Big(\sup_{t\le T}\sqrt{Y_t} > L\Big) = -\infty. \quad \text{(C.2)}$$

Then, for any $T > 0$,

$$\lim_{\beta\to 0}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log \mathsf{P}\Big(\sup_{t\le T}\big|Y_t\big| > \beta^{1/4}\Big) = -\infty.$$

Proof. Obviously Y_t solves the integral equation

$$Y_t = \mathcal{E}_t\int_0^t \mathcal{E}_s^{-1}\Big[\varepsilon\sqrt{\beta}h_3(s)\sqrt{Y_s}dM''_s + \varepsilon^2\beta h_4(s)ds\Big],$$

where $\mathcal{E}_t = \exp\big(\int_0^t[h_1(s) - \varepsilon^2 0.5h_2^2(s)]ds + \int_0^t \varepsilon h_2(s)dM'_s\big)$. Let for definiteness $|h_i| \le r$, where r is a constant. Then, with $\varepsilon \le 1$,

$$\sup_{t\le T}|\log\mathcal{E}_t| \le T(r + 0.5r^2) + \sup_{t\le T}\Big|\varepsilon\int_0^t h_2(s)dM'_s\Big|.$$

Hence the random variable $\sup_{t\le T}|\log\mathcal{E}_t|$ is bounded on the set

$$\big\{\sup_{t\le T}\big|\varepsilon\int_0^t h_2(s)dM'_s\big| \le C\big\}.$$

Moreover, it is exponentially tight in the sense that

$$\lim_{C\to\infty}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log \mathsf{P}\Big(\sup_{t\le T}|\log\mathcal{E}_t| > C\Big) = -\infty. \quad \text{(C.3)}$$

The latter is implied by

$$\lim_{C\to\infty}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log \mathsf{P}\Big(\sup_{t\le T}\big|\varepsilon\int_0^t h_2(s)dM'_s\big| > C\Big) = -\infty \quad \text{(C.4)}$$

since the martingale $N_t = \varepsilon\int_0^t h_2(s)dM'_s$ has the quadratic variation process $\langle N\rangle_t = \varepsilon^2\int_0^t h^2(s)m'(s)ds$ and, with some positive number r_1, we have $\varepsilon^2h^2(s)m'(s) \le \varepsilon^2 r_1$. Then, by taking into account that $\mathsf{P}\big(\langle N\rangle_T > \varepsilon^2 r_1 T\big) = 0$ and applying the statement (d) of Proposition A.1, we obtain $\mathsf{P}\big(\sup_{t\le T}|N_t| > C\big) \le 2e^{-C^2/(2\varepsilon^2 r_1 T)}$ providing (C.4).

Now we estimate $\sup_{t\le T}|Y_t|$ on the set $\big\{\sup_{t\le T}\big|\log\mathcal{E}_t\big|\le C\big\}$. Write

$$\sup_{t\le T}|Y_t|\le e^C Tr\varepsilon^2\beta+e^C\sup_{t\le T}\Big|\int_0^t\mathcal{E}_s^{-1}\varepsilon\sqrt{\beta}h_3(s)\sqrt{Y_s}dM''_s\Big|.$$

This upper bound and (C.2), (C.3) reduce the proof of Proposition C.1 to:

$$\lim_{\beta\to 0}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le T}\Big|\int_0^t\mathcal{E}_s^{-1}\varepsilon\sqrt{\beta}h_3(s)\sqrt{Y_s}dM''_s\Big|>\beta^{1/4},\\ \sup_{t\le T}\sqrt{Y_t}\le L,\ \sup_{t\le T}|\log\mathcal{E}_t|\le C\Big)=\infty$$

for any $C>0$ and $L>0$. Introduce the martingale

$$N''_t=\int_0^t\mathcal{E}_s^{-1}\varepsilon\sqrt{\beta}h_3(s)\sqrt{Y_s}dM''_s \text{ with } \langle N''\rangle_t=\int_0^t\mathcal{E}_s^{-2}\varepsilon^2\beta h_3^2(s)Y_s m''(s)ds$$

and denote $\mathfrak{C}=\big\{\sup_{t\le T}\sqrt{Y_t}\le L,\ \sup_{t\le T}|\log\mathcal{E}_t|\le C\big\}$. With $r_2\ge h_3^2(s)Lm''(s)$, we find that

$$\langle N''\rangle_T\le e^{2C}r_2T\varepsilon^2\beta.$$

Hence,

$$\mathsf{P}\Big(\sup_{t\le T}|N''_t|>\beta^{1/4},\mathfrak{C}\Big)=\mathsf{P}\Big(\sup_{t\le T}|N''_t|>\beta^{1/4},\ \langle N''\rangle_T\le e^{2C}r_2T\varepsilon^2\beta,\mathfrak{C}\Big)\\ \le\mathsf{P}\Big(\sup_{t\le T}|N''_t|>\beta^{1/4},\langle N''\rangle_T\le e^{2C}r_2T\varepsilon^2\beta\Big).$$

By (c) of Proposition A.1 the latter term is upper bounded by

$$2\exp\Big(\frac{\beta^{1/2}}{2e^{2C}r_2T\varepsilon^2\beta}\Big).$$

Then we obtain

$$\varlimsup_{\varepsilon\to\infty}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le T}|N''_t|>\beta^{1/4},\mathfrak{C}\Big)\le-\frac{1}{2e^{2C}r_2T\beta^{1/2}}\xrightarrow[\beta\to 0]{}-\infty.$$

□

We apply Proposition C.1 in order to prove

Lemma C.1. *For any $T>0$ and $C>0$,*

$$\lim_{\beta\to 0}\varlimsup_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\le\tau_C^\beta\wedge T}\big\|X_t^{\varepsilon,\beta}-X_t^\varepsilon\big\|>\beta^{1/4}\Big)=-\infty.$$

Proof. Recall that X_t^ε and $X_t^{\varepsilon,\beta}$ solve (1.1) and (6.6) respectively and τ_C^β is given in (6.9). Set $\triangle_t^{\varepsilon,\beta}=X^{\varepsilon,\beta}_{\tau_C^\beta\wedge t}-X^\varepsilon_{\tau_C^\beta\wedge t}$. By (1.1) and (6.6),

$$\triangle_t^{\varepsilon,\beta}=\int_0^{\tau_C^\beta\wedge t}\big(b(X^{\varepsilon,\beta}_{\tau_C^\beta\wedge s})-b(X^\varepsilon_{\tau_C^\beta\wedge s})\big)ds+\\ +\varepsilon\int_0^{\tau_C^\beta\wedge t}\big(\sigma(X^{\varepsilon,\beta}_{\tau_C^\beta\wedge s})-\sigma(X^\varepsilon_{\tau_C^\beta\wedge s})\big)dB_s+\varepsilon\sqrt{\beta}W_{\tau_C^\beta\wedge t}.$$

Due to the local Lipschitz continuity of b and σ and with $0/0 = 0$, the vector-valued and matrix-valued functions:

$$f(s) = \frac{b\big(X^{\varepsilon,\beta}_{\tau^\beta_C\wedge s}\big) - b\big(X^{\varepsilon}_{\tau^\beta_C\wedge t}\big)}{\|\triangle^{\varepsilon,\beta}_s\|} \quad \text{and} \quad g(s) = \frac{\sigma\big(X^{\varepsilon,\beta}_{\tau^\beta_C\wedge s}\big) - \sigma\big(X^{\varepsilon}_{\tau^\beta_C\wedge s}\big)}{\|\triangle^{\varepsilon,\beta}_s\|}$$

are well defined and their entries are bounded by a constant depending on C. Hence

$$\triangle^{\varepsilon,\beta}_t = \int_0^{\tau^\beta_C\wedge t} \|\triangle^{\varepsilon,\beta}_s\| f(s)ds + \varepsilon \int_0^{\tau^\beta_C\wedge t} \|\triangle^{\varepsilon,\beta}_s\| g(s)dB_s + \varepsilon\sqrt{\beta}W_{\tau^\beta_C\wedge t}.$$

Since $\|\triangle^{\varepsilon,\beta}_t\|^2 = (\triangle^{\varepsilon,\beta}_t, \triangle^{\varepsilon,\beta}_t)$, by the Itô formula, we find that

$$\begin{aligned}
\|\triangle^{\varepsilon,\beta}_t\|^2 = & \int_0^t 2\|\triangle^{\varepsilon,\beta}_s\|(\triangle^{\varepsilon,\beta}_s, f(s))ds \\
& + \varepsilon \int_0^{\tau^\beta_C\wedge t} 2\|\triangle^{\varepsilon,\beta}_s\|(\triangle^{\varepsilon,\beta}_s, g(s)dB_s) \\
& + \varepsilon\sqrt{\beta} \int_0^{\tau^\beta_C\wedge t} 2(\triangle^{\varepsilon,\beta}_s, dW_s) \\
& + \varepsilon^2 \int_0^{\tau^\beta_C\wedge t} \|\triangle^{\varepsilon,\beta}_s\|^2 \operatorname{trace}\big[g(s)g^*(s)\big]ds \\
& + \varepsilon^2\beta(\tau^\beta_C\wedge t)d.
\end{aligned} \tag{C.5}$$

Now, by letting $\phi(s) = \frac{2(\triangle^{\varepsilon,\beta}_s, f(s))}{\|\triangle^{\varepsilon,\beta}_s\|}$ and $d\widehat{B}_s = \frac{2(\triangle^{\varepsilon,\beta}_s, g(s)dB_s)}{\|\triangle^{\varepsilon,\beta}_s\|}$, we rewrite (C.5) as:

$$\begin{aligned}
\|\triangle^{\varepsilon,\beta}_t\|^2 = & \int_0^{\tau^\beta_C\wedge t} \|\triangle^{\varepsilon,\beta}_s\|^2\Big(\phi(s) + \varepsilon^2 \operatorname{trace}[g(s)g^*(s)]\Big)ds \\
& + \varepsilon \int_0^{\tau^\beta_C\wedge t} \|\triangle^{\varepsilon,\beta}_s\|^2 d\widehat{B}_s + \varepsilon\sqrt{\beta} \int_0^{\tau^\beta_C\wedge t} \|\triangle^{\varepsilon,\beta}_s\| \frac{2(\triangle^{\varepsilon,\beta}_s, dW_s)}{\|\triangle^{\varepsilon,\beta}_s\|} \\
& + \varepsilon^2\beta(\tau^\beta_C\wedge t)d.
\end{aligned} \tag{C.6}$$

With the notations

- $Y_t = \|\triangle^{\varepsilon,\beta}_t\|^2$
- $h_1(s) = I_{\{\tau^\beta_C\leq s\}}\{\phi(s) + \varepsilon^2 \operatorname{trace}[g(s)g^*(s)]\}$
- $h_2(s) \equiv 1$
- $h_4(s) = I_{\{\tau^\beta_C\leq s\}}d$
- $M'_t = \widehat{B}_t$, $m'(s) = \dfrac{4(\triangle^{\varepsilon,\beta}_s, g(s)g^*(s)\triangle^{\varepsilon,\beta}_s)}{\|\triangle^{\varepsilon,\beta}_s\|^2}$
- $M''_t = \int_0^{\tau^\beta_C\wedge t} 2\dfrac{(\triangle^{\varepsilon,\beta}_s, dW_s)}{\|\triangle^{\varepsilon,\beta}_s\|}$, $m''(s) \equiv 4$,

the equation (C.6) is in the form of (C.1). Since $h_i(s)$, $i = 1, \ldots, 4$ are bounded and $\sqrt{Y}_t \equiv \|X^{\varepsilon,\beta}_{\tau^\beta_C\wedge t} - X^{\varepsilon}_{\tau^\beta_C\wedge t}\| \leq \|X^{\varepsilon,\beta}_{\tau^\beta_C\wedge t}\| + \|X^{\varepsilon}_{\tau^\beta_C\wedge t}\| \leq 2C$, i.e., (C.2) holds too, the statement of the lemma follows from Proposition C.1. □

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